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Time-Dependent First Integrals, Nonlinear Dynamical Systems, and Numerical Integration

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Many nonlinear dynamical systems expressed as autonomous systems of first-order ordinary differential equations admit first integrals and explicitly time-dependent first integrals. Under numerical integration these first integrals should be preserved. We discuss this case for explicitly time-dependent first integrals.

KEY WORDS: dynamical systems; first integrals; numerical integration.

Consider the autonomous system of first-order ordinary differential equations

$$\frac{d\mathbf{u}}{dt} = V(\mathbf{u}) \tag{1}$$

where $V_j : \mathbf{R}^n \to \mathbf{R}$ are analytic functions. Many of these dynamical systems admit first integrals $I(\mathbf{u}(t))$ and explicitly time-dependent first integrals $I(\mathbf{u}(t), t)$ (Steeb, 2005). The later case we often find in dynamical systems with chaotic behavior, for example the Lorenz model and the Rikitake two-disc dynamo (Steeb, 1982). That $I(\mathbf{u}(t))$ is a first integral is expressed as $dI/dt = 0$. Using the analytic vector field (Steeb, 1996)

$$V = \sum_{j=1}^{n} V_j(\mathbf{u}) \frac{\partial}{\partial u_j}$$

we can express this condition as $L_V I(\mathbf{u}) = 0$, where L_V denotes the Lie derivative. Thus from $dI/dt = 0$ we can write

$$\sum_{j=1}^{n} \frac{\partial I}{\partial u_j} V_j(\mathbf{u}) = 0.$$

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McLaren and Quispel (2004) (see also McLachlan *et al.*, 1999) discussed the case of integral-preserving integrators if system (1) admits a first integral $I(\mathbf{u}(t))$. It can be shown that (1) can be written as

$$\frac{d\mathbf{u}}{dt} = S(\mathbf{u})\nabla I(\mathbf{u})$$

where $S(\mathbf{u})$ is a skew-symmetric $n \times n$ matrix and ∇ denotes the gradient. Note that ∇I is considered as a column vector. The matrix $S(\mathbf{u})$ is given by

$$S(\mathbf{u}) = \frac{1}{|\nabla I|^2}(V(\nabla I)^{\mathrm{T}} - (\nabla I)V^{\mathrm{T}})$$

where $^{\mathrm{T}}$ denotes transpose. Obviously we have to assume that $|\nabla I|$ is nonvanishing. An integral preserving discrete version of this is given by (McLachlan *et al.*, 1999; McLaren and Quispel, 2004)

$$\frac{\mathbf{u}' - \mathbf{u}}{\tau} = \bar{S}(\mathbf{u}, \mathbf{u}', \tau)\overline{\nabla} I(\mathbf{u}, \mathbf{u}')$$

where τ is the step length, $\mathbf{u}$, $\mathbf{u}'$ denote $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$, respectively. The matrix $\bar{S}$ is a skew symmetric matrix satisfying for consistency

$$\bar{S}(\mathbf{u}, \mathbf{u}', \tau) = S(\mathbf{u}) + O(\tau).$$

The general discrete gradient $\overline{\nabla} I$ is defined by

$$(\mathbf{u}' - \mathbf{u}) \cdot \overline{\nabla} I(\mathbf{u}', \mathbf{u}) := I(\mathbf{u}') - I(\mathbf{u}).$$

As an example consider a Lotka–Volterra model with three species $(u_1, u_2, u_3 > 0)$

$$\frac{du_1}{dt} = u_1 u_2 - u_1 u_3$$
$$\frac{du_2}{dt} = u_2 u_3 - u_1 u_2$$
$$\frac{du_3}{dt} = u_3 u_1 - u_2 u_3.$$

It describes the interaction between three species, where species 1 feeds on species 2, species 2 feeds on species 3, and species 3 feeds on species 1. The model is of interest since it admits two first integrals, namely $I_1(\mathbf{u}) = u_1 + u_2 + u_3$ and $I_2(\mathbf{u}) = u_1 u_2 u_3$. The fixed points of this system is the manifold $\{(u_1, u_2, u_3) : u_1 = u_2 = u_3\}$. From the constants of motion $u_1 + u_2 + u_3 = C_1$ and $u_1 u_2 u_3 = C_2$, where $C_1 > 0$, $C_2 > 0$ and from stability analysis we find that the system admits closed orbits as solutions. Using the approach given above we have to decide which of the two first integrals we use for the discretization.

For explicitly time-dependent first integrals we extended the autonomous system (1) to the autonomous system in $\mathbf{R}^{n+1}$

$$\frac{d\mathbf{u}}{d\epsilon} = V(\mathbf{u})$$
$$\frac{dt}{d\epsilon} = 1$$

where $t(\epsilon = 0) = 0$. Then we have the vector field in $\mathbf{R}^{n+1}$

$$W = V + \frac{\partial}{\partial t}$$

and the definition for the explicitly first integral $dI(\mathbf{u}(t), t)/dt = 0$ can be written as $L_W I(\mathbf{u}, t) = 0$. Thus the method described above can be extended to explicitly time-dependent first integrals. As an example with explicitly time-dependent first integrals consider the Lorenz model

$$\frac{du_1}{dt} = \sigma(u_2 - u_1)$$
$$\frac{du_2}{dt} = -u_2 - u_1(u_3 - r)$$
$$\frac{du_3}{dt} = u_1 u_2 - b u_3.$$

For example for $b = 2\sigma$ and r arbitrary we find the explicitly time-dependent first integral

$$I(\mathbf{u}(t), t) = \left(u_1^2 - 2\sigma u_3\right)\exp(2\sigma t).$$

Other explicitly time-dependent first integrals for the Lorenz model are given by Kuś (1983). These first integrals can be found using an ansatz given by Steeb (1982) and the Carleman embedding (Kowalski and Steeb, 1990). Another example of a dynamical system with explicitly time-dependent first integrals is the Rikitake-two-disc dynamo

$$\frac{du_1}{dt} = -r u_1 + u_3 u_2$$
$$\frac{du_2}{dt} = -r u_2 + (u_3 - a) u_1$$
$$\frac{du_3}{dt} = 1 - u_1 u_2.$$

In this case we have the explicitly time-dependent first integral for $a = 0$ and r arbitrary (Steeb, 1982)

$$I(\mathbf{u}(t), t) = (u_1^2 - u_2^2)\exp(2rt).$$

There are also systems which admit two explicitly time-dependent first integrals. For example

$$\frac{du_1}{dt} = cu_1 + c_{23}u_2u_3$$
$$\frac{du_2}{dt} = cu_2 + c_{13}u_1u_3$$
$$\frac{du_3}{dt} = cu_3 + c_{12}u_1u_2$$

with the explicitly time-dependent first integrals

$$I_1(\mathbf{u}(t), t) = \frac{1}{2}\left(c_{13}u_1^2 - c_{23}u_2^2\right)e^{-2ct}, \qquad I_2(\mathbf{u}(t), t) = \frac{1}{2}\left(c_{12}u_1^2 - c_{23}u_3^2\right)e^{-2ct}.$$

Using the approach for the discretization given above we again have to decide which of the explicitly time-dependent first integrals we are applying.

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